Confining model for the inverse analytical coupling constant in QCD

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Summary. — The behavior of the strong coupling constant $\alpha_s(q^2)$ in Quantum Chromodynamics (QCD) as a function of energy is characterized by a divergence at low values of q^2, which describes the confinement phenomenon typical of the strong interaction. Theoretically, it is possible to reproduce the running of the coupling constant using a perturbative series expansion in the limit $\alpha_s \to 0$. However, when the strong coupling constant becomes too large, perturbation theory loses its validity, and in this regime, the Analytic Perturbation Theory (APT) framework becomes relevant. This work aims to develop a model capable of reproducing confinement by exploiting the analyticity introduced by APT. In particular, by working with the Inverse Coupling Constant $\varepsilon_s(q^2) = 1/\alpha_s(q^2)$, the confinement phenomenon is translated into the vanishing of ε_s at $q^2 = 0$.

1. – Introduction

The energy dependence of the strong coupling constant $\alpha_s(q^2)$ in Quantum Chromodynamics (QCD) is characterized by a divergence at low values of q^2, which accounts for the confinement phenomenon inherent to the strong interaction. At the theoretical level, the behavior of the coupling constant can be reproduced through a perturbative series expansion in the limit $\alpha_s \to 0$. In the leading logarithmic approximation, the expression of $\alpha_s(q^2)$ is:

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0}\frac{1}{\ln(Q^2/\Lambda^2)}, \tag{1}$$

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with $Q^2 = -q^2$ (spacelike), $\beta_0 = 11 - 2n_f/3$ and n_f number of active quarks flavour. When the strong coupling constant becomes too large perturbative models develop unphysical divergences, known as *ghosts*, which prevent the correct reproduction of the coupling behavior below those ghosts. At one-loop, the strong interaction has only a simple pole: $Q^2 = \Lambda^2 = (300\ \mathrm{MeV})^2$ (Landau's pole). Standard perturbation theory does not hold if $Q^2 \ll \Lambda^2$.

2. – Analytic Perturbation Theory (APT)

Analytic Perturbation Theory (APT), by exploiting the principle of causality which requires the analyticity of propagators in the complex q^2 plane cut along the negative real axis, allows for the analytic continuation of the function $\alpha_s(q^2)$ through the use of the Källén-Lehmann representation (KL)

$$\alpha_s(q^2) = \int_0^\infty \frac{\rho(\sigma)}{\sigma + q^2} d\sigma, \tag{2}$$

where $\rho(\sigma)$ is the *spectral density* and corresponds to the imaginary part of $\alpha_s(q^2)$ calculated on the lower edge of the physical cut, i.e.,

$$\rho(\sigma) = \lim_{\epsilon \to 0^+} \mathrm{Im}[\alpha_s(-\sigma - i\epsilon)]. \tag{3}$$

By calculating the spectral density $\rho(\sigma)$ from eq. (3) using eq. (1) and then solving the integral of eq. (2), we obtain

$$[\alpha_s(Q^2)]_{an} = \frac{4\pi}{\beta_0}\left[\frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2}\right]. \tag{4}$$

Even though this formulation regularizes α_s at $Q^2 = \Lambda^2$, it yields a fixed finite value in the infrared region (IR), not reproducing the divergent confinement behavior. In particular, its value at the origin $Q^2 = 0$ is $[\alpha_s(0)]_{an} = 4\pi/\beta_0$ and it is independent of the order of the loop expansion. This behavior is called *infrared stability.*

3. – Inverse Coupling Constant ICC

This work aims to find a model capable of reproducing the confinement against the *infrared stability*, exploiting the analyticity applied by the APT. In particular it is useful working with the Inverse Coupling Constant (ICC) $\varepsilon_s(Q^2) = 1/\alpha_s(Q^2)$. In fact the confinement for the ICC is translated in

$$\lim_{Q^2 \to 0^+} [\varepsilon_s(Q^2)]_{an} = 0. \tag{5}$$

This means that if we apply the KL to the ICC, the spectral density, which is the imaginary part of perturbative $\varepsilon_s(Q^2)$, also has to be zero in the same limit. So it is possible to reproduce confinement by introducing a regularizing function $r(\sigma = Q^2)$ in the spectral density to achieve eq. (5). Moreover, the regularizing function must also not spoil the correct perturbative ultraviolet (UV) limit. Therefore, this function can be chosen from the set of arbitrary continuous functions fulfilling the conditions

$$r(\sigma)\begin{cases}\xrightarrow[\sigma\to 0^+]{} 0\\ \xrightarrow[\sigma\to+\infty]{} 1\end{cases}. \tag{6}$$

The ICC is expanded perturbatively as:

$$4\pi\frac{\partial\varepsilon_s}{\partial\ln(\mu^2)}=\beta_0+\frac{\beta_1}{4\pi\varepsilon_s}+\cdots=\sum_{n=0}^{\infty}\frac{\beta_n}{(4\pi\varepsilon_s)^n}. \tag{7}$$

The solution of eq. (7) truncated at the leading ultraviolet (UV) behavior of 2-loop order is

$$\varepsilon_s(Q^2)=\underbrace{K_0\ln\left(\frac{Q^2}{\Lambda^2}\right)}_{\varepsilon_s^{(0)}(Q^2)}+\underbrace{K_1\ln\left(\frac{K_0}{K_1}\ln\left(\frac{Q^2}{\Lambda^2}\right)\right)}_{\varepsilon_s^{(1)}(Q^2)}, \tag{8}$$

with $\beta_0=11-2n_f/3$, $\beta_1=102-38n_f/3$, $K_0=\beta_0/(4\pi)$ and $K_1=\beta_1/(4\pi\beta_0)$. The 1-loop term is $\varepsilon_s^{(0)}(Q^2)$, while $\varepsilon_s^{(1)}(Q^2)$ is the 2-loop correction. The expressions of the two spectral densities are obtained from

$$\rho^{(0,1)}(\sigma)=\lim_{\alpha\to 0^+}\mathrm{Im}[\varepsilon_s^{(0,1)}(-\sigma-i\alpha)], \tag{9}$$

whence, using eq. (8):

$$\rho^{(0)}(\sigma)=-\pi K_0, \tag{10a}$$

$$\rho^{(1)}(\sigma)=-K_1\mathrm{arccotan}\Big(\frac{\ln(\sigma/\Lambda^2)}{\pi}\Big). \tag{10b}$$

It follows that the KL representation becomes a dispersion relation subtracted at $t=\Lambda^2$ (1):

$$[\bar{\varepsilon}_s^{(0,1)}(t)]_{an}=[\bar{\varepsilon}_s^{(0,1)}(\Lambda^2)]_{an}+\frac{(\Lambda^2-t)}{\pi}\int_0^{\infty}d\sigma\frac{\bar{\rho}^{(0,1)}(\sigma)}{(\sigma+t)(\sigma+\Lambda^2)}, \tag{11}$$

where $t=Q^2$ is the spacelike Mandelstam variable and the regularized spectral densities are

$$\bar{\rho}^{(0,1)}(\sigma)=\rho^{(0,1)}(\sigma)\cdot r(\sigma). \tag{12}$$

(1) The dispersion relation subtracted at $t=\Lambda^2$ is required, otherwise the integral would be divergent for any complex value of t. The subtraction parameter $[\bar{\varepsilon}_s^{(0,1)}(\Lambda^2)]_{an}$ is then fixed by requiring the vanishing at $Q^2=0$ of the functions of eq. (11).

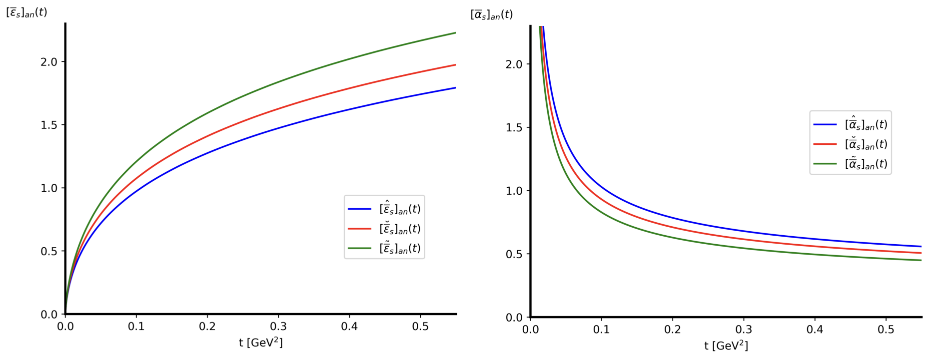


Fig. 1. – Left: The ICC $[\bar{\varepsilon}_s(t)]_{an}$, with $p = 0.8$, $\Lambda = 300$ MeV and $n_f = 5$. Right: Four-momentum distribution for $[\bar{\alpha}_s(t)]_{an}$, with $p = 0.8$, $\Lambda = 300$ MeV and $n_f = 5$.

In ref. [1] we propose these three parameterizations for the regularizing function $r(\sigma) = r(\sigma, p, \Lambda(n_f))$ ensuring the correct high- and low-momentum behaviors of eq. (6):

$$\hat{r}_p(\sigma) = \frac{1}{1 + (\Lambda^2/\sigma)^p}; \tag{13a}$$

$$\breve{r}_p(\sigma) = \frac{1}{(1 + \Lambda^2/\sigma)^p}; \tag{13b}$$

$$\tilde{r}_p(\sigma) = \left(1 - e^{-\sigma/\Lambda^2}\right)^p, \tag{13c}$$

where $p \in (0, 1)$ and $\sigma \in \mathbb{R}^+$. To simplify the reading, we present here just the results obtained by using the eq. (13c).

The confining analytical expressions for ICC at 1-loop and 2-loop are:

$$[\tilde{\bar{\varepsilon}}_s^{(0)}(z)]_{an} = [\varepsilon_s^{(0)}(z)]_{an} + K_0\gamma \sum_{k=1}^{\infty} (-1)^k \binom{p}{k} [e^{kz}\Gamma(0, kz) + \ln(k)], \tag{14a}$$

$$[\tilde{\bar{\varepsilon}}_s^{(1)}(z)]_{an} = [\varepsilon_s^{(1)}(z)]_{an} + \sum_{k=1}^{\infty} (-1)^k \binom{p}{k} \tilde{I}_k(z), \tag{14b}$$

in which we have $z = t/\Lambda^2$, for eq. (14a) the Euler-Mascheroni constant $\gamma \approx 0.5772$, the *Exponential integral function* $\Gamma(0, x) = E_1(x)$(2) and for eq. (14b)

$$\tilde{I}_k(z) = \frac{1}{\pi} \int_0^{\infty} dx \frac{\rho^{(1)}(x)(1 - e^{-kz})}{x(x+1)} + \frac{1-z}{\pi} \int_0^{\infty} dx \frac{\rho^{(1)}(x) \cdot e^{-kz}}{(x+z)(x+1)}. \tag{15}$$

The details regarding the calculations of this model and the other two parameterizations are given in ref. [1]. Adding eqs. (14) we obtain the total confining analytical expression $[\tilde{\bar{\varepsilon}}_s(t)]_{an} = [\tilde{\bar{\varepsilon}}_s^{(0)}(t)]_{an} + [\tilde{\bar{\varepsilon}}_s^{(1)}(t)]_{an}$, from which it is possible to calculate $[\tilde{\bar{\alpha}}_s(t)]_{an} =$

(2) *Exponential integral function*: $\Gamma(0, x) = \int_x^{\infty} \frac{e^{-t}}{t} dt = E_1(x)$.

		$[\tilde{\tilde{\alpha}}_s]_{an}$		
Λ \ p	0.5	0.6	0.7	0.8
0.2	0.10058	0.10406	0.10674	0.10887
0.3	0.10616	0.11004	0.11304	0.11544
0.4	0.11052	0.11473	0.11800	0.12062

Fig. 2. – . Best values of the parameter Λ at fixed values of the parameter p for the model $[\tilde{\tilde{\alpha}}_s(t)]_{an}$.

$1/[\tilde{\tilde{\varepsilon}}_s(t)]_{an}$.The fig. 1 shows the momentum distributions according to our model for all the three regularizing function parametrizations, where $n_f = 5$, $p = 0.8$ and $\Lambda = 300$ MeV.

4. – Numerical results

The models were characterized by analyzing their behavior as a function of the model parameters p and $\Lambda(n_f)$ [1]. We outline the results of (13c) only.

4˙1. *Fitting the experimental data of $\alpha_s(q^2)$ measured by JADE, LEPII, and CMS.* – First, a fit was performed on the experimental α_s data measured by JADE, LEPII, and CMS [2]. The fit was carried out with p as a free parameter and $\Lambda = 350$ MeV and $n_f = 5$ fixed. The following value was obtained for p:

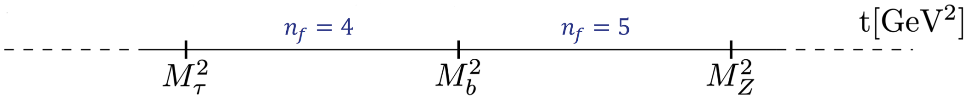
$$p = 0.80 \pm 0.05 \qquad (\chi^2 = 0.724). \tag{16}$$

The theoretical curve is shown on the left in fig. 4.

4˙2. *Calculation of the coupling constant at M_Z and M_τ.* – Subsequently, the value of α_s at the mass of the Z boson $M_Z \approx 91.19$ GeV was calculated, obtaining the values reported in the table of fig. 2 for the model in eq. (13c). In fact, our values $\alpha_s(M_Z)$ are compatible with the world average value [3]:

$$\alpha_s^{\mathrm{PDG}}(M_Z^2) = 0.1180 \pm 0.009, \tag{17}$$

for all three regularizing function parameterizations. For all values of p and Λ presented in 2, the difference between the model prediction and the PDG value (17) is less than



n_f	2	3	4	5	6
Λ (GeV)	0.634	0.579	0.49	0.35	0.144

Fig. 3. – Values of the momentum scale Λ as a function of the number of active flavours n_f.

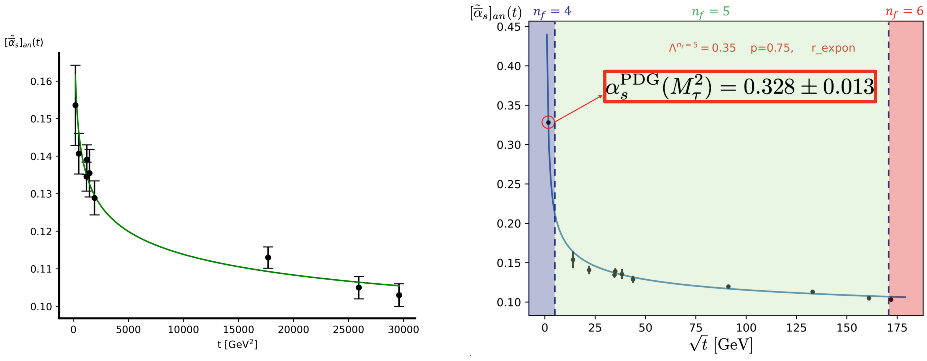


Fig. 4. – Left: The running coupling constant $[\tilde{\alpha}_s(t)]_{an}$, with $\Lambda = 350$ MeV and the optimal values of p (16). Right: The running coupling constant $[\tilde{\alpha}_s(t)]_{an}$ with continuity condition with $\Lambda = 350$ and $p = 0.75$.

10% [1].

From the study of the parameters in the perturbative region, initial values were extracted to be used for validations at lower energy scales. To test the validity of the models at lower energies, it is necessary to account for the dependence of Λ on n_f. The analysis was extended down to an energy scale corresponding to the τ lepton mass M_τ. However, in order to compute $\alpha_s(M_\tau)$, it is necessary to determine $\Lambda^{(n_f=4)}$ from $\Lambda^{(n_f=5)} = 350$ MeV. This dependence is obtained by imposing the continuity condition: $[\tilde{\alpha}_s^{(n_f-1)}(m_f^2)]_{an} = [\tilde{\alpha}_s(t)^{(n_f)}(m_f^2)]_{an}$ [4]. In fig. 3 are shown the tabulated values of Λ depending on the number of active flavours. Therefore, it was possible to reproduce the value of α_s also at $Q^2 = M_\tau^2$, as shown on the right in the fig. 4.

5. – Conclusions

The procedure for defining an analytic expression of the ICC as a function of the four-momentum squared, having the desired IR behavior, was achieved by exploiting the APT approach to introduce a parametric regularizing function which ensured the vanishing of the ICC as $Q^2 \to 0$. Three possible parameterizations were considered for such a regularizing function, depended on the same pair of parameters, the power p and the momentum scale Λ. A first attempt to characterize the model was presented by reproducing experimental values of the strong coupling constant at fixed energy scales, with the goal of progressively lowering the scale and entering the non-perturbative region $(\alpha_s(4M_\pi^2))$. Furthermore, future developments include the computation of physical observables that depend on the strong coupling constant α_s, such as the *Hadronic decay rate of the τ lepton (R_τ)*, and the *Hadronic contribution to the anomalous magnetic moment of the muon (a_μ)*.

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